

## PHYS-3203 Homework 2 Due 22 Jan 2020

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

### 1. Sliding on a Cycloid

Consider a cycloidal track (like we found for the brachistochrone)

$$x = a(\theta - \sin \theta) , \quad y = a(1 - \cos \theta) \quad (1)$$

with  $y$  increasing downward. The coordinate  $\theta$  extends from  $\theta = 0$  at the left of the track  $x = 0, y = 0$  to  $\theta = 2\pi$  at the right  $x = 2\pi a, y = 0$ , and the lowest point of the track is  $x = a\pi, y = 2a$  at  $\theta = \pi$ .

- (a) Write the Lagrangian and Euler-Lagrange equation for an object sliding frictionlessly on the cycloid in terms of the generalized coordinate  $\theta$ . Show that the equation of motion is that of a harmonic oscillator for  $\theta = \pi + u$  where  $|u| \ll 1$  (ie, small amplitude motion around the lowest point of the cycloid).
- (b) To learn more about motion on the cycloid, try a different generalized coordinate. Define

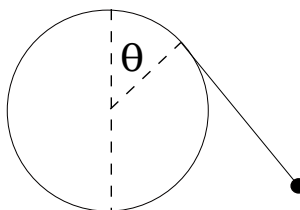
$$s = \int_{\pi}^{\theta} d\theta' \sqrt{\left(\frac{dx}{d\theta'}\right)^2 + \left(\frac{dy}{d\theta'}\right)^2} , \quad (2)$$

so  $|s|$  is the distance traveled from the lowest point along the cycloid. Write the Lagrangian in terms of  $s$ . *Hint:* Use the Pythagorean theorem to find the kinetic energy and then show  $y = 2a - s^2/8a$  to get the potential energy.

- (c) Write the Lagrangian for a simple harmonic oscillator of frequency  $\omega$ . By comparison with your result from part (b), show that motion on the cycloid is simple harmonic and give the frequency (which is independent of amplitude). You do not need to find the Euler-Lagrange equations.

### 2. Pendulum Wrapping a Disk *inspired by TM*

A mass  $m$  is attached to one end of a string of length  $l$ , with the other end attached to the uppermost point of a fixed vertical disk of radius  $R$  ( $\pi R < l < 3\pi R/2$ ), as in the figure below. The string wraps around the edge of the disk in a clockwise direction. In the following, use the angle  $\theta$  where the string leaves the disk as a generalized coordinate.



- (a) Assuming the motion lies in the  $xy$  plane with the origin at the center of the disk, find the  $x$  (horizontal) and  $y$  (vertical) position of the mass as a function of  $\theta$ .
- (b) Write the Lagrangian for this system in terms of  $\theta$ . Find the equation of motion for the pendulum.
- (c) Find the equilibrium position  $\theta_0$  where the potential energy is minimized.
- (d) Write  $\theta = \theta_0 + \phi$ , where  $\phi$  is a small angle. Expand the Lagrangian to second order in  $\phi$  (including  $\dot{\phi}$ ) and find the frequency of small oscillations.