PHYS-4601 Homework 15 Due 12 Feb 2015

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Quantum Information & the Density Operator**

   In the class notes (and reading), we briefly mentioned that quantum information should depend only on the state of the system and not on the measurements performed but should still reproduce the Shannon entropy. In this problem, we will briefly explore the quantum *von Neumann entropy.*

   Suppose you have a friend in a laboratory that produces a quantum system in some specific state, but you don't know what state it is in. You know that there is a probabilty P_n that your friend has made state $|\psi_n\rangle$. From your perspective, the system is in a *mixed state*, which is described by the classical probability of each quantum state. We can describe this mixed state by a *density operator*

   $$\rho \equiv \sum_n P_n \, |\psi_n\rangle\langle\psi_n| \ , \tag{1}$$

   which represents your ignorance of the true quantum state of the system.

   (a) Suppose the system is the spin of an electron, and there is a 50% probability each that your friend has produced the spin either up along z or up along x. Write the density matrix first in terms of the states $|\uparrow_z\rangle, |\uparrow_x\rangle$ and then the basis states $|\uparrow_z\rangle, |\downarrow_z\rangle$.

   The von Neumann entropy is defined as

   $$S \equiv -\sum_i \langle e_i| \left(\rho \log_2 \rho\right) |e_i\rangle \ , \tag{2}$$

   where the sum is over a complete basis of states $|e_i\rangle$. We can understand this by writing $\rho = 1 + (\rho - 1)$ and treating $\log_2 \rho$ as a power series in $(\rho - 1)$.

   (b) A *pure state* is a single quantum state, so that one probability $P_n = 1$ and all the others vanish. For a pure state $|\psi\rangle$, the density operator is $\rho = |\psi\rangle\langle\psi|$. Show that the von Neumann entropy (2) is zero for a pure state. *Hint:* Show that a pure state $|\psi\rangle$ has $(\rho - 1)|\psi\rangle = 0$.

   (c) Finally, consider that an electron might be prepared with probability p spin up along z (that is, state $|\uparrow_z\rangle$) and probability $1-p$ spin down along z (in $|\downarrow_z\rangle$). Show that the von Neumann entropy is equal to the Shannon entropy $-p\log_2 p - (1-p)\log_2(1-p)$ for this system.

2. **White Dwarfs** *based on Griffiths 5.35*

   White dwarfs are old, dead stars that don't support nuclear fusion any more. Instead, electron degeneracy pressure keeps them from collapsing into black holes. In this problem, you'll work out the white dwarf radius by finding the minimum energy. *Note:* You will need equation [5.45] from Griffiths.

   (a) Imagine that the white dwarf is a sphere of uniform density. Caculate the gravitational potential energy V in terms of its radius R, total mass M, and Newton's constant G. *Hint*: To do this, work out the potential energy of a shell of thickness dr at radius r, which comes from the sphere of matter inside that shell. Then integrate from $r = 0 \to R$. You should find $V = -3GM^2/5R$.

(b) Now treat the electrons as a free electron gas. Write their total energy (which is kinetic) in terms of M, R, the electron mass m, the nucleon mass m_N, the number of electrons per nucleon q, and $\hbar$. Then add this to the result of part (a) to get the total energy and find the radius R where the energy is minimized.

(c) Assuming $q \approx 1/2$ (the average nucleus is a helium nucleus), find the radius (in km) of a white dwarf with the mass of the sun. You will need $G = 7\times 10^{-11}$ m^3/kg/s^2, $\hbar = 1\times 10^{-34}$ Js, $m = 9 \times 10^{-31}$ kg, $m_N = 2 \times 10^{-27}$ kg, and $M = 2 \times 10^{30}$ kg.

(d) A neutron star is an extremely dense star where all the matter is neutrons, which is supported against collaps by the degeneracy pressure of the neutrons. In this case, $q = 1$. Find the ratio of the radius of a neutron star to the radius of a white dwarf of the same mass (assume $q \approx 1$ for the white dwarf), both in terms of physical constants and as a pure number to 1 significant digit.

3. **Bloch's Theorem**

In class, we stated Bloch's theorem as saying that any stationary state wavefunction of a periodic potential $V(x + a) = V(x)$ can be written to satisfy the condition

$$\psi(x + a) = e^{iKa}\psi(x) \ (K \text{ real}) \ . \tag{3}$$

Alternately, Bloch's theorem can state that any stationary state wavefunction of the same potential can be written as

$$\psi(x) = e^{iKx}u(x) \text{ where } u(x + a) = u(x) \ . \tag{4}$$

(a) Show that these two formulations are equivalent (that is, show that if ψ satisfies (3) then it satisfies (4) and vice-versa).

(b) If you write $u(x) = \sum_q c_q e^{iqx}$ as a Fourier series, what are the allowed values of q? Therefore, what are the allowed values of the momentum for a wavefunction with a given K?