PHYS-4601 Homework 11 Due 8 Jan 2015

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Rotations** *parts of Griffiths 4.56*

   (a) Think back to our earlier problems on the translation operator. Argue that $\exp[i\varphi L_z/\hbar]$ is a rotation around the z axis by showing that

   $$e^{i\varphi L_z/\hbar} \cdot \psi(\phi) = \psi(\phi + \varphi) \tag{1}$$

   for any angular wavefunction $\psi(\phi)$ that can be written as a Taylor series around ϕ. *Hint*: This should be basically identical to what you did for the translation operator if you use the identification that $L_z = -i\hbar\partial/\partial\phi$.

   As a result, the angular momentum operators are called the *generators* of rotations. In general, $\hat{n} \cdot \vec{L}/\hbar$ generates rotations around the unit vector $\hat{n}$. Furthermore, the rotations of spinors are generated by the spin angular momentum operators.

   (b) What is the 2×2 matrix corresponding to a rotation of 2π around the z axis for spin 1/2? How does it compare to what you expected?

   (c) Construct the 2×2 matrix corresponding to a rotation of π around the x axis for spin 1/2. Show that it takes the S_z eigenstate $|+\rangle$ into $|-\rangle$.

2. **Angular Momentum in Hydrogen** *related to Griffiths 4.35*

   Consider an electron in a hydrogen atom with $\ell = 1$ orbital angular momentum (and of course spin $s = 1/2$).

   (a) What are the allowed total angular momentum quantum numbers j for the electron?

   (b) As stated in class, protons also have spin $s = 1/2$; we can say that the "spin" of this hydrogen atom is the total angular momentum of all the components. What are the allowed total spin quantum numbers of this atom? How many complete sets of states are there for each of those total spins? Argue that you have found the correct total number of states.

3. **Spin Interactions**

   (a) Two spin 1/2 particles are fixed in position but have interacting spins. Their Hamiltonian is

   $$H = J\vec{S}^{(1)} \cdot \vec{S}^{(2)} \tag{2}$$

   for some constant J. Here $S^{(i)}$ is the spin operator of the ith particle. Find the energy eigenvalues of this system, their degeneracies, and the corresponding eigenstates. *Hint*: You will want to work in terms of the total spin quantum numbers.

   (b) The two spins have the same gyromagnetic ratio γ. In the presence of a magnetic field, the Hamiltonian becomes

   $$H = J\vec{S}^{(1)} \cdot \vec{S}^{(2)} - \gamma\vec{B} \cdot \left(\vec{S}^{(1)} + \vec{S}^{(2)}\right) \ . \tag{3}$$

   Now find the energy eigenvalues and their degeneracies. You may take $\vec{B}$ to lie along the z direction.

4. **Center of Mass Frame and Reduced Mass**

In class, we treat the hydrogen atom as if it is an electron moving around a stationary proton. Of course, that can't be, since it violates conservation of momentum. What happens, of course, is that the proton hardly moves in the center of mass rest frame. However, it turns out that we can always describe a system of two particles in terms of a single particle. In this problem, consider two particles of masses m_1 and m_2.

(a) In quantum mechanics, the kinetic energy is given by a Laplacian operator. Consider the 1D case for simplicity. Then the kinetic Hamiltonian is

$$H = -\frac{\hbar^2}{2m_1}\frac{d^2}{dx_1^2} - \frac{\hbar^2}{2m_2}\frac{d^2}{dx_2^2} , \quad (4)$$

where x_1 is the first particle's position and x_2 is the second particle's position. Show that this kinetic Hamiltonian can be written as

$$H = -\frac{\hbar^2}{2M}\frac{d^2}{dX^2} - \frac{\hbar^2}{2\mu}\frac{d^2}{dx^2} , \quad (5)$$

where $M = m_1 + m_2$ is the total mass, $\mu = m_1 m_2/M$ is the reduced mass, $X = (m_1 x_1 + m_2 x_2)/M$ is the center of mass position, and $x = x_1 - x_2$ is the relative position.

The proof is essentially the same for the 3D Laplacian, and we then set the center of mass momentum to zero by choice of reference frame. Therefore, when we study the hydrogen atom, we are really using the reduced mass of the electron, which is nearly the electron mass because the proton is so much heavier than the electron.

(b) Imagine an atom made of an electron and a deuteron (nucleus made of a proton and neutron); this is a deuterium atom. Deuterium atoms are exactly like hydrogen atoms (in terms of energy eigenvalues) except the proton mass is replaced by the deuteron mass. Find the fractional difference in ground state energies $(E_D - E_H)/E_H$, where E_D is the deuterium ground state energy and E_H is the hydrogen ground state energy. First, give your answer in terms of the electron, proton, and deuteron masses to $\mathcal{O}(m_e/m_p, m_e/m_d)$ and then numerically using $m_e = 0.511$ MeV$/c^2$, $m_p = 938$ MeV$/c^2$, $m_d = 1876$ MeV$/c^2$ (give your answer to three significant digits).

5. **MRI Physics** *Inspired by a question by Horbatsch*

Consider a spin-1/2 proton with gyromagnetic ratio γ in the presence of a magnetic field

$$\vec{B} = B_0\hat{z} + B_1\cos(\omega t)\hat{x} - B_1\sin(\omega t)\hat{y} \quad (6)$$

at its fixed position. This is a magnetic field with a fixed z component and another component rotating in the x, y plane.

(a) Write the Hamiltonian in terms of spin operators and then as a matrix in the eigenbasis of the S_z operator.

(b) It is actually possible to find the full time-dependent state for this system. If the spin is up at $t = 0$, the solution of the time-dependent Schrödinger equation is

$$\begin{aligned} \langle +|\Psi(t)\rangle &= e^{i\omega t/2}\left[\cos(\alpha t/2) - i\frac{(\omega - \gamma B_0)}{\alpha}\sin(\alpha t/2)\right] \\ \langle -|\Psi(t)\rangle &= ie^{-i\omega t/2}\frac{\gamma B_1}{\alpha}\sin(\alpha t/2) \end{aligned} \quad (7)$$

with $\alpha = \sqrt{\gamma^2 B_1^2 + (\omega - \gamma B_0)^2}$. Use Maple to verify that (7) solves the Schrödinger equation. Input the Schrödinger equation and initial conditions as a list of equations and then the solution above as another list. Then use the `odetest` function in Maple to check that (7) solves the time-dependent Schrödinger equation. Include a copy of your Maple code. You may want to use the Maple help to learn how to use `odetest`.

(c) Use (7) to find the transition probability from spin up ($|+\rangle$) to spin down ($|-\rangle$). Find the conditions that this probability is one.