Quantum Mechanics II PHYS-4601
Second In-Class Test

Dr. Andrew Frey

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Instructions:

- Do not turn over until instructed.
- You will have 75 minutes to complete this test.
- No electronic devices or hardcopy notes are allowed.
- INSTRUCTIONS REGARDING THE QUESTIONS WILL GO HERE.
- **Only the lined pages of your exam book will be graded. Use the blank pages for scratch work only.**

Useful Formulae:

- Schrödinger Equation
  - time-dependent and position-basis time-independent

$$i\hbar\frac{d}{dt}|\Psi\rangle = H|\Psi\rangle \ , \quad \langle\vec{x}|H|\psi\rangle = -\frac{\hbar^2}{2m}\nabla^2\psi(\vec{x}) + V(\vec{x})\psi(\vec{x}) = E\psi(\vec{x})$$

  - Radial equation

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right]u = Eu \ , \quad u(r) = rR(r)$$

- Angular momentum
  - Commutation relations (ϵ_{ijk} is the antisymmetric tensor):

$$[L_i, L_j] = i\hbar\sum_k \epsilon_{ijk}L_k \ , \quad [L_z, L_\pm] = \pm\hbar L_\pm \text{ for } L_\pm = L_x \pm iL_y \quad (\text{and for } \vec{L}\to\vec{S})$$

  - $s = 1/2$ spin operators in the S_z eigenbasis are $\vec{S} = (\hbar/2)\vec{\sigma}$, with Pauli matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \ , \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \ , \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ .$$

  - The "total" quantum number j for two added angular momenta of quantum numbers j_1 and j_2 is $j = j_1 + j_2, j_1 + j_2 - 1, \cdots |j_1 - j_2|$ (one multiplet of each value)
- Hydrogen
  - States are denoted $|n, \ell, m, m_s\rangle$ or $|n, j, \ell, m_j\rangle$ (recall that $s = 1/2$ always for electrons).

- Bohr radius $a = 4\pi\epsilon_0\hbar^2/me^2$ and energy $E_n = -(\hbar^2/2ma^2)(1/n^2) = -13.6\text{ eV}/n^2$
- Spatial wavefunction (see below for Laguerre polynomials and spherical harmonics)

$$\psi_{n\ell m}(\vec{x}) \equiv \langle\vec{x}|n,\ell,m\rangle = \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-\ell-1)!}{2n[(n+\ell)!]^3}} e^{-r/na}\left(\frac{2r}{na}\right)^\ell L^{2\ell+1}_{n-\ell-1}\left(\frac{2r}{na}\right) Y^m_\ell(\theta,\phi)$$

- Spherical harmonics and associated Legendre functions

$$Y^m_\ell = (-1)^m\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}e^{im\phi}P^m_\ell(\cos\theta)\ (m\geq 0)\ ,\quad Y^{-m}_\ell = (-1)^m(Y^m_\ell)^*$$

$$P^m_\ell(x) = (1-x^2)^{m/2}\left(\frac{d}{dx}\right)^m P_\ell(x)\ ,\quad P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell$$

- Associated Laguerre polynomials

$$L^p_{q-p}(x) = (-1)^p\left(\frac{d}{dx}\right)^p L_q(x)\ ,\quad L_q(x) = e^x\left(\frac{d}{dx}\right)^q (e^{-x}x^q)$$

TABLE 4.3: The first few spherical harmonics, $Y_l^m(\theta,\phi)$.

$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2}$	$Y_2^{\pm 2} = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta e^{\pm 2i\phi}$
$Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta$	$Y_3^0 = \left(\frac{7}{16\pi}\right)^{1/2}(5\cos^3\theta - 3\cos\theta)$
$Y_1^{\pm 1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}$	$Y_3^{\pm 1} = \mp\left(\frac{21}{64\pi}\right)^{1/2}\sin\theta(5\cos^2\theta - 1)e^{\pm i\phi}$
$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta - 1)$	$Y_3^{\pm 2} = \left(\frac{105}{32\pi}\right)^{1/2}\sin^2\theta\cos\theta e^{\pm 2i\phi}$
$Y_2^{\pm 1} = \mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta e^{\pm i\phi}$	$Y_3^{\pm 3} = \mp\left(\frac{35}{64\pi}\right)^{1/2}\sin^3\theta e^{\pm 3i\phi}$

TABLE 4.6: Some associated Laguerre polynomials, $L^p_{q-p}(x)$.

$L_0^0 = 1$	$L_0^2 = 2$
$L_1^0 = -x + 1$	$L_1^2 = -6x + 18$
$L_2^0 = x^2 - 4x + 2$	$L_2^2 = 12x^2 - 96x + 144$
$L_0^1 = 1$	$L_0^3 = 6$
$L_1^1 = -2x + 4$	$L_1^3 = -24x + 96$
$L_2^1 = 3x^2 - 18x + 18$	$L_2^3 = 60x^2 - 600x + 1200$

- Possibly useful integrals
  - Gaussian integral

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}}e^{b^2/4a}$$

  where a, b can be complex as long as $\text{Re}\, a > 0$
  - Exponential integrals

$$\int_0^\infty dx\, x^p e^{-x/b} = p! b^{p+1}$$